Politechnika Warszawska | Wydział Mechaniczny Energetyki i Lotnictwa | Zakład Wytrzymałości Materiałów i Konstrukcji

Wykład 8

Charakterystyki geometryczne figur płaskich

Wskaźniki charakteryzujące przekrój pręta (figurę płaską)

A – pole

Momenty statyczne:

$$S_\eta = \int_A \zeta dA$$

$$S_\zeta = \int_A \eta \, dA$$

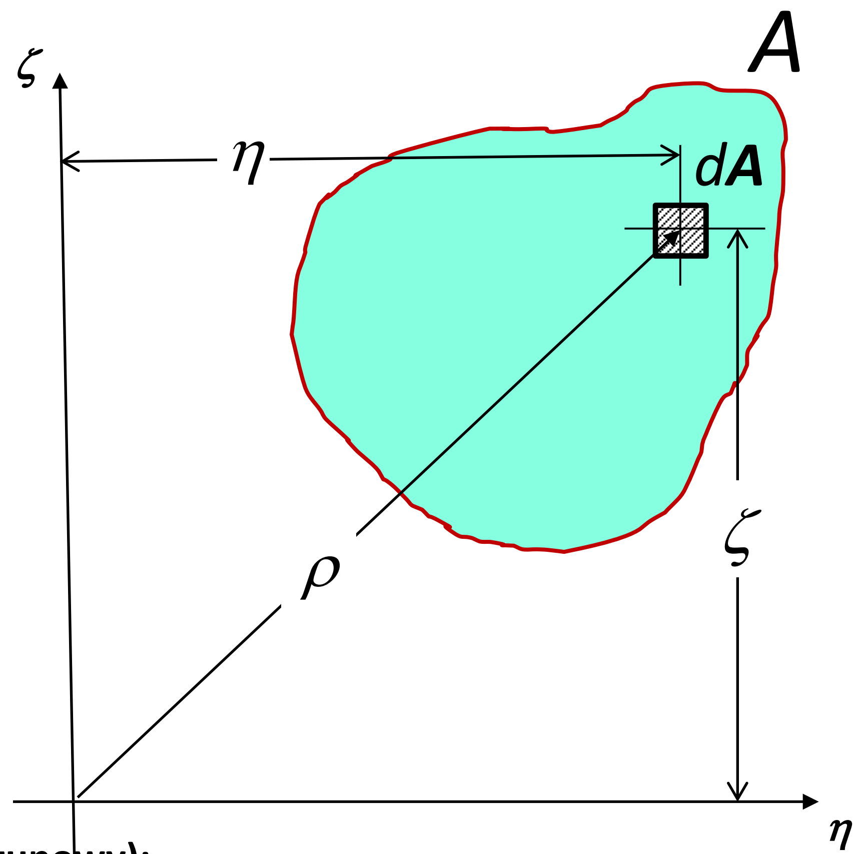
Momenty bezwładności:

$$J_\eta = \int_A \zeta^2 dA$$

$$J_\zeta = \int_A \eta^2 dA$$

Moment dewiacji (odśrodkowy):

$$J_{\eta\zeta} = \int_A \eta \, \zeta dA$$



Momenty bezwładności względem punktu (biegunowy):

$$J_0 = \int_A \rho^2 \, dA = J_\eta + J_\zeta$$

Centralne osie bezwładności

(y, z) to osie *centralne (przechodzą przez środek ciężkości figury)* i równoległe do osi układu *(η,ζ)*

Wtedy momenty statyczne:

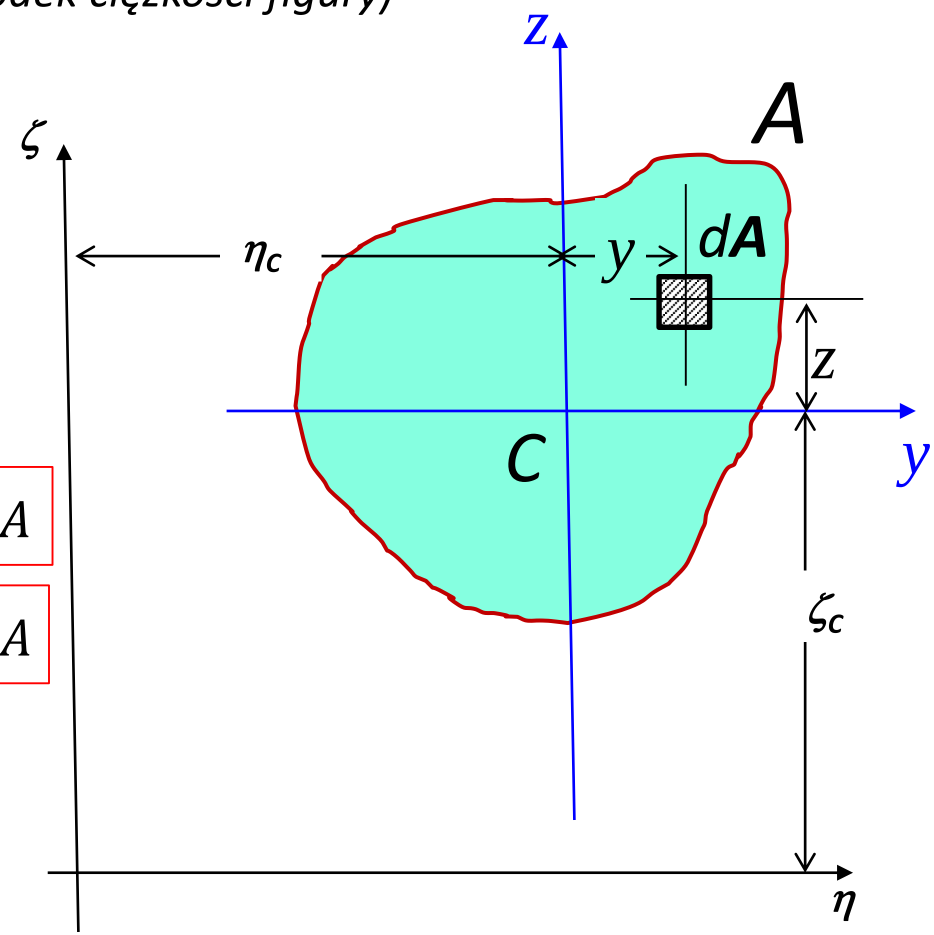
$$\boxed{S_y = \int_A z\, dA = 0}$$

$$\boxed{S_z = \int_A y\, dA = 0}$$

Wtedy współrzędne środka ciężkości:

$$\boxed{S_\eta = \int_A \zeta dA = \zeta_C\, A} \Rightarrow \boxed{\zeta_C = \frac{1}{A}\int_A \zeta dA}$$

$$\boxed{S_\zeta = \int_A \eta\, dA = \eta_C A} \Rightarrow \boxed{\eta_C = \frac{1}{A}\int_A \eta dA}$$



Centralne momenty bezwładności:

$$\boxed{J_y = \int_A z^2 dA} \qquad \boxed{J_z = \int_A y^2 dA}$$

Centralny moment dewiacji (odśrodkowy):

$$\boxed{J_{yz} = \int_A y\, z\, dA}$$

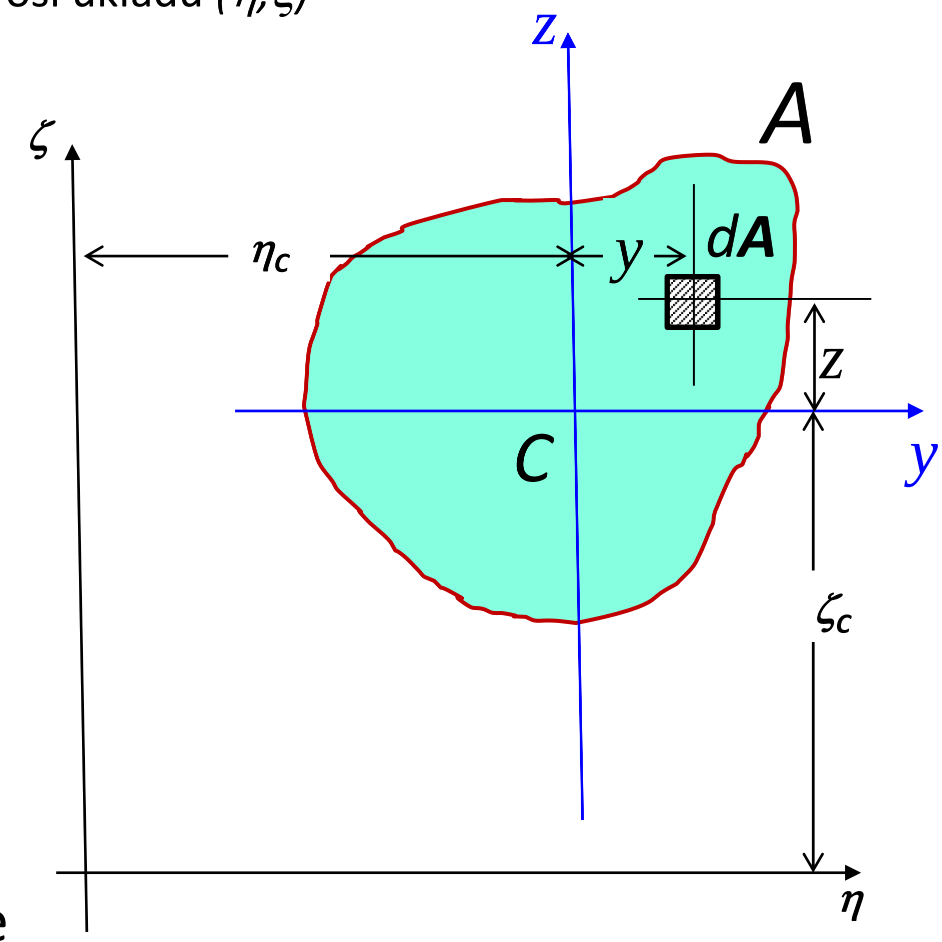
Twierdzenie Steinera

Jeśli *(y, z)* to osie *centralne* i równoległe do osi układu *(η,ζ)*

$$\boxed{J_\eta = J_y + A\,\zeta_C{}^2}$$

$$\boxed{J_\zeta = J_z + A\,\eta_C{}^2}$$

$$\boxed{J_{\eta\zeta} = J_{yz} + A\,\eta_C\,\zeta_C}$$



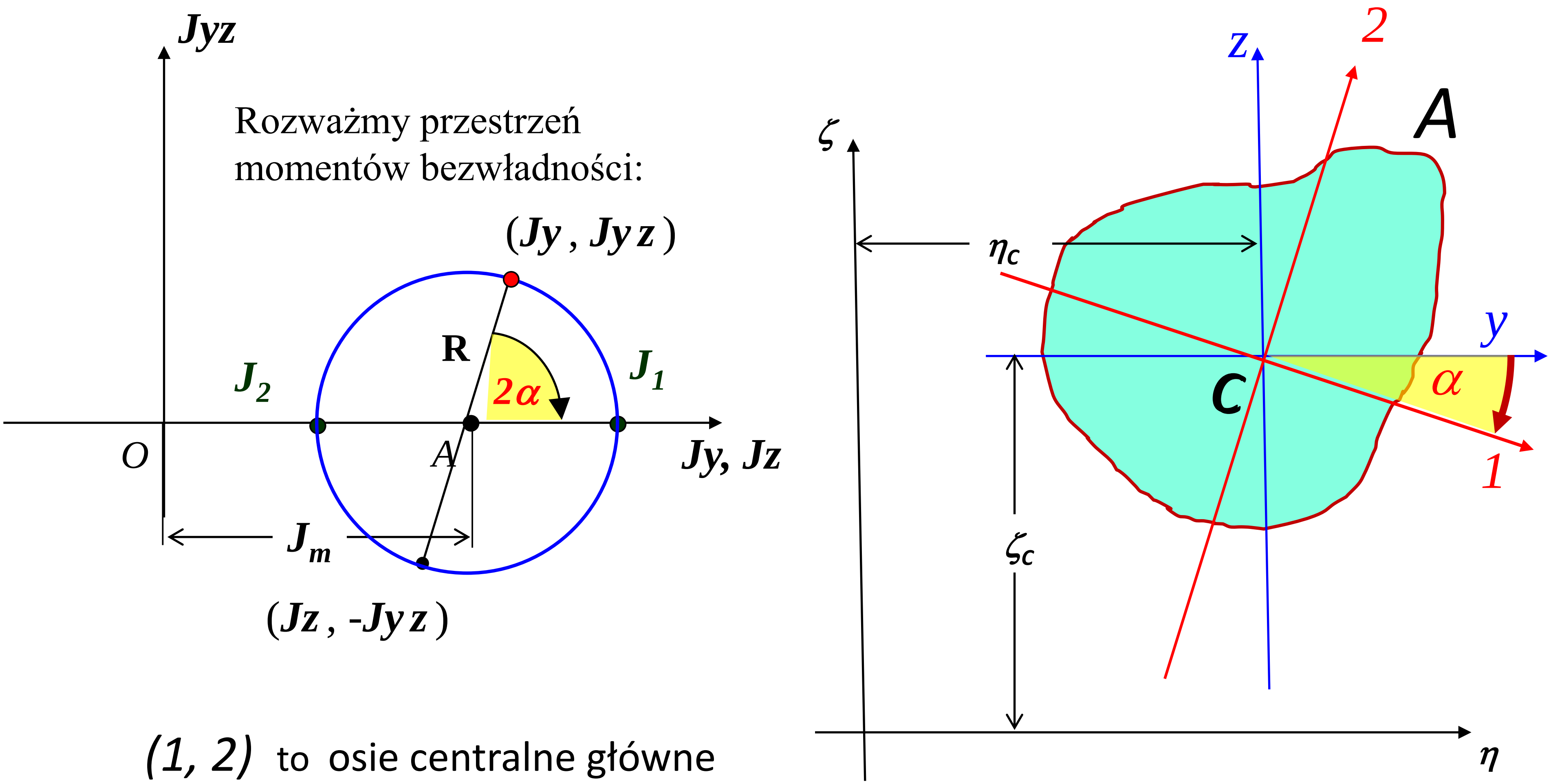
Jeśli:

$J_{\eta\zeta} = 0$ ➔ osie główne

$S_y = 0$; $S_z = 0$ ➔ osie centralne

$J_{yz} = 0$ ➔ Osie główne i centralne

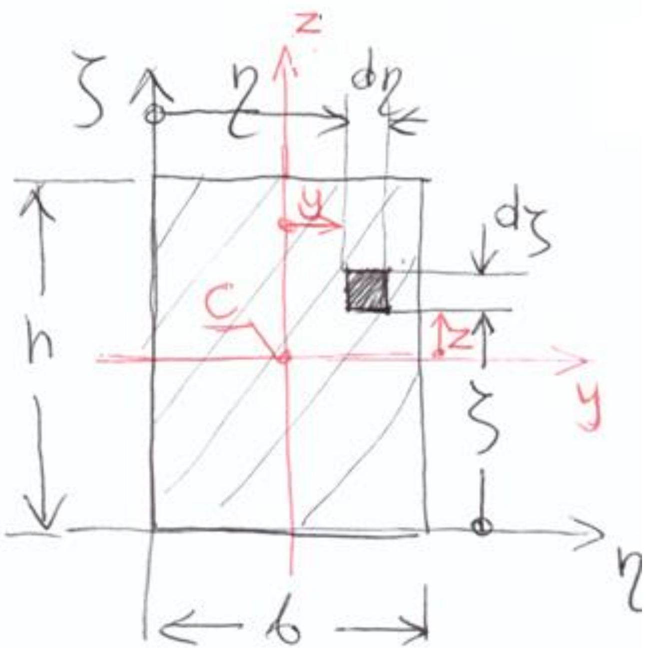
Wyznaczenie centralnych głównych momentów bezwładności



(1, 2) to osie centralne główne

Zadanie 8.1

Zad. 1. Wyznaczyć centralne główne momenty bezwładności



Wyznaczamy SC:

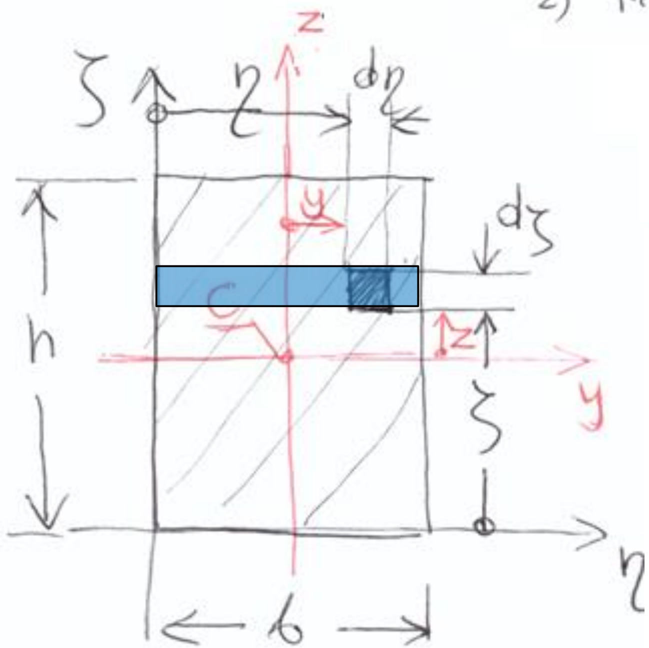
$$S_\eta = \int_A \zeta \, dA = \zeta_0 \cdot A$$

$$\zeta_c = \frac{1}{A} \int_A \zeta \, dA = \frac{1}{b \cdot h} \int_0^h \int_0^b \zeta \cdot d\eta \, d\zeta$$

$$= \frac{1}{b \cdot h} \cdot b \int_0^h \zeta \, d\zeta = \frac{1}{h} \frac{1}{2} \zeta^2 \Big|_0^h = \frac{h}{2}$$

Podobnie $\boxed{\eta_c = \frac{b}{2}}$

Zadanie 8.1 (cd)



2) Momenty bezwładności względem osi centralnych

$$J_y = \int_A z^2 dA = \int_{-h/2}^{h/2} z^2 b\,dz = b \cdot \frac{1}{3} z^3 \Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{1}{3} b\left(\frac{h^3}{8} - \left(-\frac{h^3}{8}\right)\right)$$

$$\boxed{J_y = \frac{bh^3}{12}}$$ Podobnie $$\boxed{J_z = \frac{h \cdot b^3}{12}}$$

3) Moment dewiacji

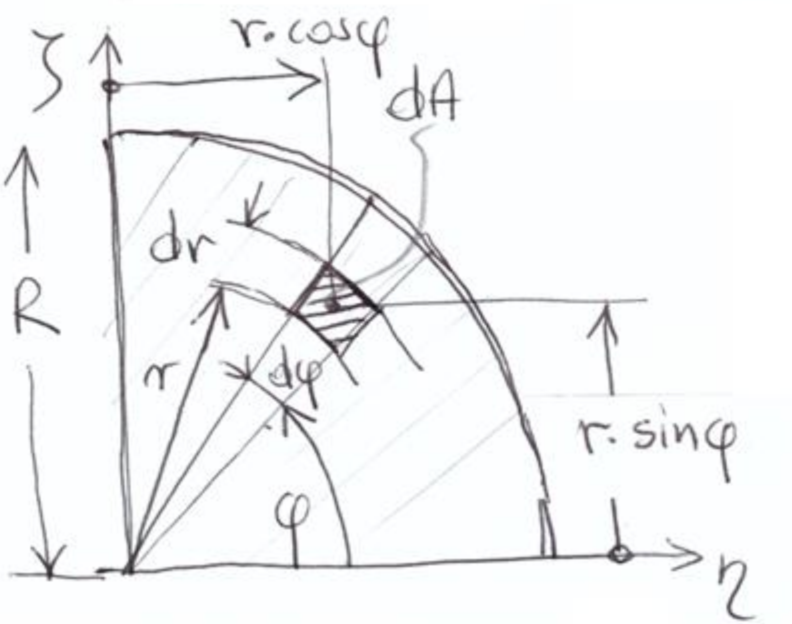
$$J_{yz} = \int_A yz\,dA = \int_{-\frac{h}{2}}^{h/2} \int_{-\frac{b}{2}}^{b/2} yz\,dy\,dz = \int_{-\frac{h}{2}}^{h/2} z\left(\frac{1}{2} y^2 \Big|_{-\frac{b}{2}}^{\frac{b}{2}}\right) dz$$

$$\boxed{J_{yz} = 0}$$ A więc osie główne i centralne

Wniosek: Osie symetrii są osiami głównymi

Zadanie 8.2

Zad. 2. Wyznaczyć centralne, główne momenty bezwładności



1) Pole $\boxed{A = \frac{\pi R^2}{4}}$

$$\boxed{dA = dr \cdot r \cdot d\varphi}$$

2) środek ciężkości

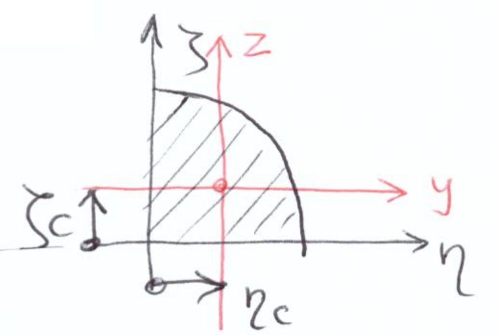
$$\boxed{\zeta_c = \frac{1}{A}\int_A \zeta \cdot dA}$$

$$\zeta_c = \frac{1}{A}\int_0^R \int_0^{\pi/2} r\sin\varphi \, dr\, r d\varphi = \frac{1}{A}\int_0^R r^2 \int_0^{\pi/2} \sin\varphi \, d\varphi \, dr = \frac{1}{A}\int_0^R r^2 \left(-\cos\varphi\Big|_0^{\pi/2}\right) dr$$

$$= \frac{1}{A}\int_0^R r^2 dr = \frac{4}{3}\frac{R^3}{\pi R^2}$$

$$\boxed{\zeta_c = \frac{4}{3}\frac{R}{\pi}} \qquad \boxed{\eta_c = \frac{4}{3}\frac{R}{\pi}}$$

Zadanie 8.2 (c.d.)



3) Momenty bezwładności

$$J_\eta = \int_A \zeta^2 dA = \int_0^R \int_0^{\pi/2} r^2 \sin^2\varphi \; r\,d\varphi\, dr =$$

$$= \int_0^R r^3 \int_0^{\pi/2} \sin^2\varphi\, d\varphi\; dr = \int_0^R r^3 \int_0^{\pi/2} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\varphi\right) d\varphi\, dr$$

$$= \int_0^R r^3 \tfrac{1}{2}\left(\varphi\Big|_0^{\pi/2} - \tfrac{1}{2}\sin 2\varphi\Big|_0^{\pi/2}\right) dr = \frac{\pi}{4}\int_0^R r^3 dr = \frac{\pi}{16} R^4$$

$$\boxed{J_\eta = \frac{\pi}{16} R^4} \qquad \boxed{J_\zeta = \frac{\pi}{16} R^4}$$

4) Moment dewiacji

$$J_{\eta\zeta} = \int_A \zeta\eta\, dA = \int_0^R \int_0^{\pi/2} r\sin\varphi \cdot r\cos\varphi \; r\,d\varphi\, dr =$$

$$= \int_0^R r^3 \int_0^{\pi/2} \tfrac{1}{2}\sin 2\varphi\, d\varphi\; dr = \frac{1}{4}\int_0^R r^3 \Big(\underbrace{-\cos 2\varphi\Big|_0^{\frac{\pi}{2}}}_{=2}\Big) dr$$

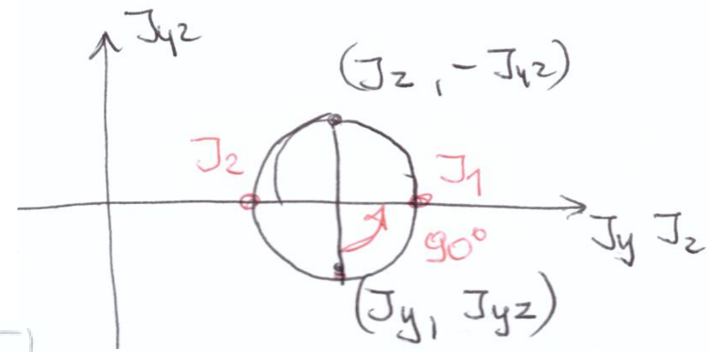
$$\boxed{J_{\eta\zeta} = \frac{1}{2}\int_0^R r^3 dr = \frac{1}{8} R^4}$$

5) Momenty centralne

Tw. Steinera: $\boxed{J_\eta = J_y + A\zeta_c^2}$

$$\boxed{J_y = J_\eta - A\zeta_c^2} = \frac{\pi}{16}R^4 - \frac{\pi R^2}{4}\left(\frac{4}{3}\frac{R}{\pi}\right)^2$$

$$\boxed{J_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)R^4} \approx \boxed{0.05488\,R^4} = \boxed{J_z}$$

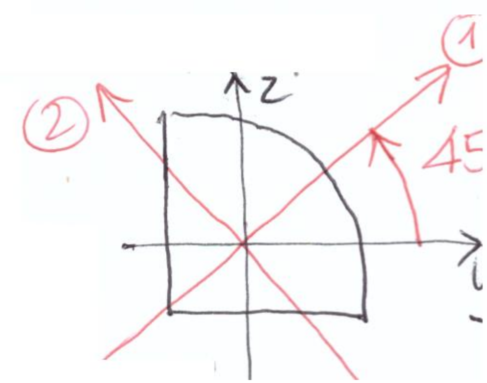


$$\boxed{J_1 = 0.07138\,R^4 \qquad J_2 = 0.03838\,R^4}$$

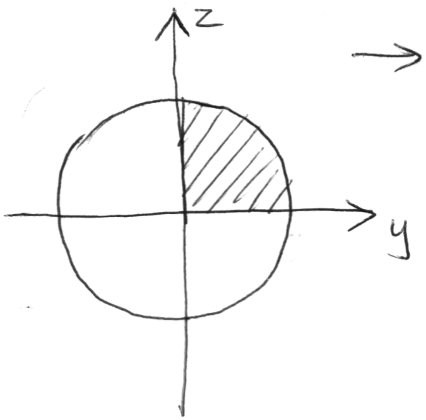
$$\boxed{J_{yz} = J_{\eta\zeta} - A\zeta_c\eta_c}$$

$$J_{yz} = \frac{1}{8}R^4 - \frac{\pi R^2}{4}\left(\frac{4}{3}\frac{R}{\pi}\right)\left(\frac{4}{3}\frac{R}{\pi}\right)$$

$$\boxed{J_{yz} = \left(\frac{1}{8} - \frac{4}{9\pi}\right)R^4} \approx \boxed{-0.0165\,R^4}$$



Popularne przekroje – warto znać na pamięć te wzory!

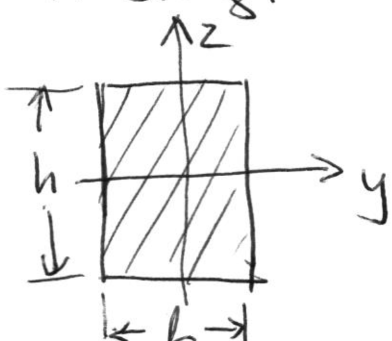


→ Pełne koło

$$\boxed{J_y = 4 \times \frac{\pi}{16} R^4 = \frac{\pi}{4} R^4}$$

$$\boxed{J_y = \frac{\pi D^4}{64}}$$

→ Prostokąt

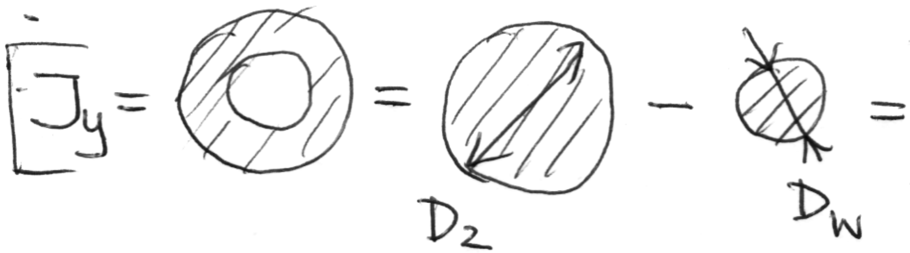


$$\boxed{J_y = \frac{b h^3}{12}}$$

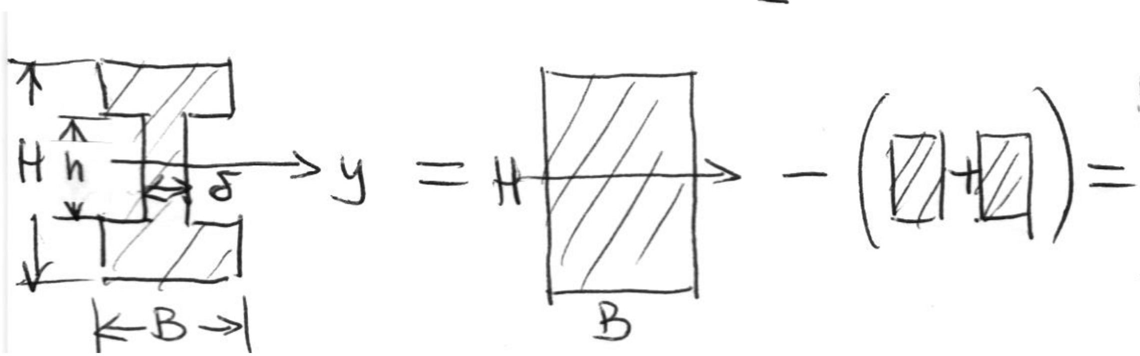
$$\boxed{J_z = \frac{h b^3}{12}}$$

Chwyty stosowane w obliczeniach

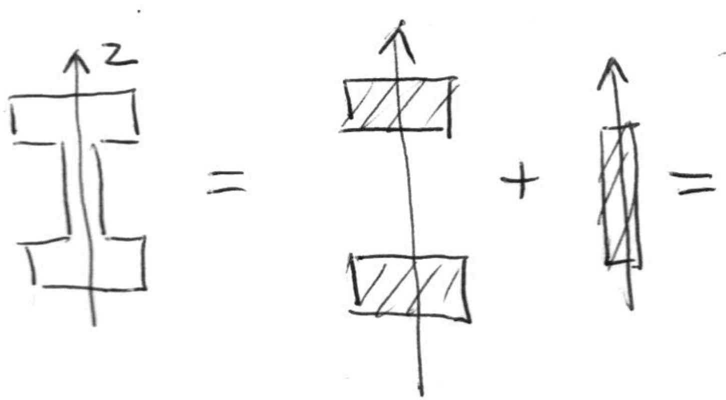
CHWYTY



$$\boxed{\frac{\pi}{64}\left(D_2^4 - D_W^4\right)}$$

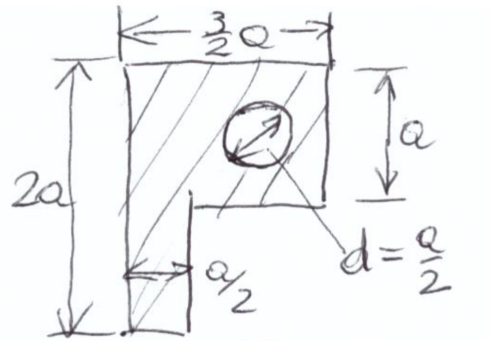


$$\boxed{\frac{BH^3}{12} - \frac{(B-\delta)h^3}{12}}$$

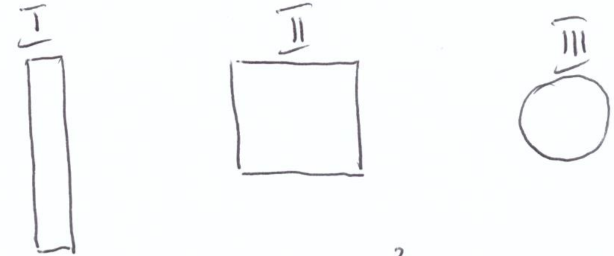


$$\boxed{\frac{(H-h)B^3}{12} + \frac{h\cdot\delta^3}{12}}$$

Zadanie 8.2



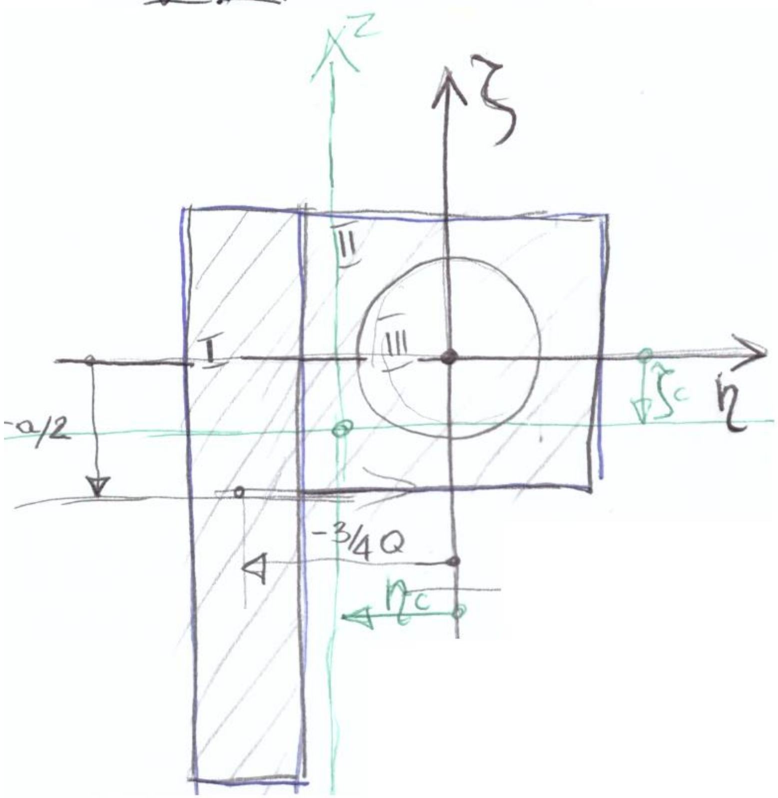
$$\boxed{A = A_{I} + A_{II} - A_{III}}$$



$$A_{I} = a^2 \qquad A_{II} = a^2 \qquad A_{III} = \frac{\pi a^2}{16}$$

$$\eta_{cI} = -\frac{3}{4}a \qquad \eta_{cII} = 0 \qquad \eta_{cIII} = 0$$

$$\zeta_{cI} = -\frac{1}{2}a \qquad \zeta_{cII} = 0 \qquad \zeta_{cIII} = 0$$



$$J_{ycI} = \frac{\frac{a}{2}(2a)^3}{12} \qquad J_{ycII} = \frac{a^4}{12} \qquad J_{ycIII} = \frac{\pi\left(\frac{a}{2}\right)^4}{64}$$

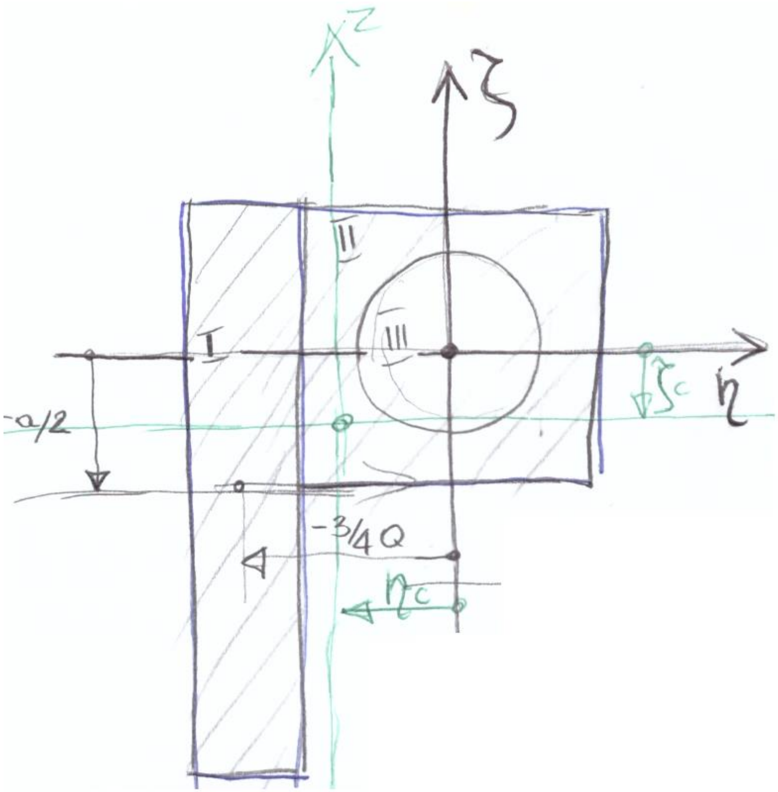
$$J_{zcI} = \frac{2a\left(\frac{a}{2}\right)^3}{12} \qquad J_{zcII} = \frac{a^4}{12} \qquad J_{zcIII} = \frac{\pi\left(\frac{a}{2}\right)^4}{64}$$

$$\eta_c = \frac{1}{A}\int_A \eta\, dA = \frac{1}{A}\left[A_{I}\cdot\eta_{cI} + A_{II}\cdot\eta_{cII} - A_{III}\cdot\eta_{cIII}\right]$$

$$\eta_c = -\frac{12a}{32-\pi} \cong \boxed{-0.416a}$$

$$\zeta_c = \frac{1}{A}\int_A \zeta\, dA = \frac{1}{A}\left[A_{I}\cdot\zeta_{cI} + A_{II}\cdot\zeta_{cII} - A_{III}\cdot\zeta_{cIII}\right]$$

$$\zeta_c = -\frac{8a}{32-\pi} \cong \boxed{-0.277a}$$



$$J_y = J_{y_{I}} + J_{y_{II}} - J_{y_{III}} = J_{yc_{I}} + A_{I}\cdot(\zeta_{c_{I}} - \zeta_c)^2 +$$

$$+ J_{yc_{II}} + A_{II}(\zeta_{c_{II}} - \zeta_c)^2 - J_{yc_{III}} - A_{III}(\zeta_{c_{III}} - \zeta_c)^2 =$$

$$= \frac{\frac{a}{2}(2a)^3}{12} + a^2\cdot\left(-\frac{a}{2} - \left(-\frac{8a}{32-\pi}\right)\right)^2 + \frac{a^4}{12} + a^2\left(0 - \left(-\frac{8a}{32-\pi}\right)\right)^2 +$$

$$- \frac{\pi\left(\frac{a}{2}\right)^4}{64} - \frac{\pi\left(\frac{a}{2}\right)^2}{4}\left(0 - \left(-\frac{8a}{32-\pi}\right)\right)^2 \cong \boxed{0.525\, a^4}$$

$$J_z = J_{z_{I}} + J_{z_{II}} - J_{z_{III}} = J_{zc_{I}} + A_{I}(\eta_{c_{I}} - \eta_c)^2 +$$

$$+ J_{zc_{II}} + A_{II}(\eta_{c_{II}} - \eta_c)^2 - J_{zc_{III}} - A_{III}(\eta_{c_{III}} - \eta_c)^2 =$$

$$= \frac{2a\left(\frac{a}{2}\right)^3}{12} + a^2\left(-\frac{3}{4}a - \left(-\frac{12a}{32-\pi}\right)\right)^2 + \frac{a^4}{12} + a^2\left(0 - \left(-\frac{12a}{32-\pi}\right)\right)^2 +$$

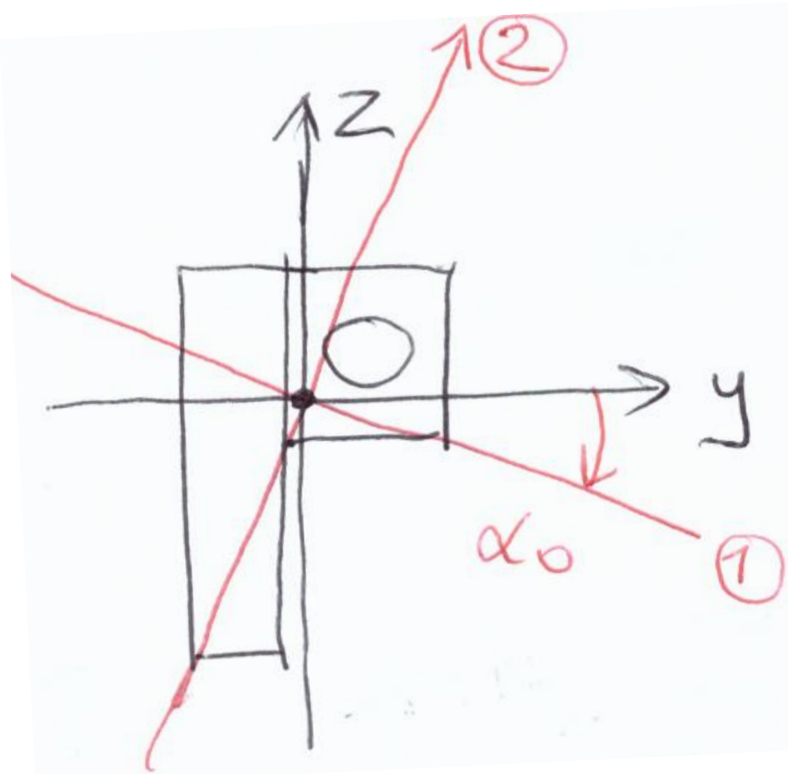
$$- \frac{\pi\left(\frac{a}{2}\right)^4}{64} - \frac{\pi\left(\frac{a}{2}\right)^2}{4}\left(0 - \left(-\frac{12a}{32-\pi}\right)\right)^2 \cong \boxed{0.3517\, a^4}$$

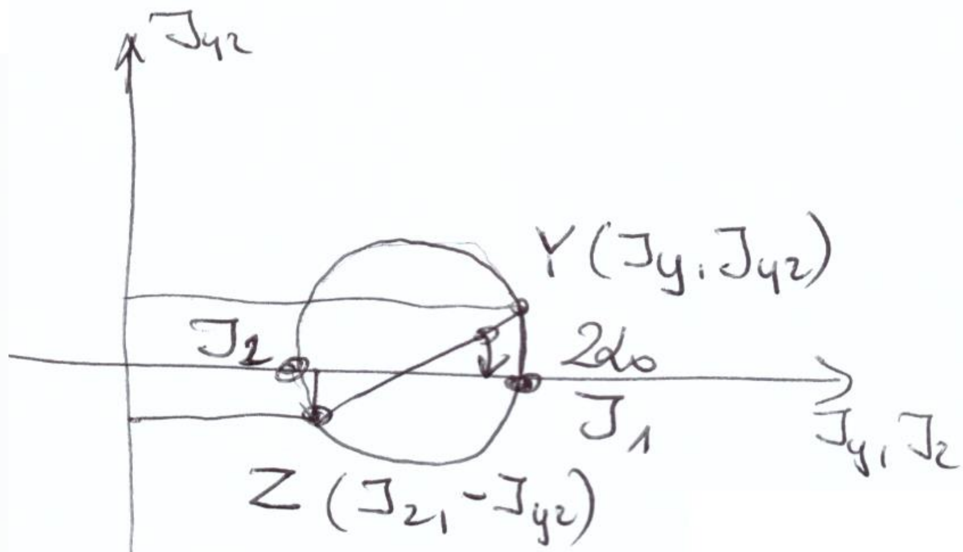
$$J_{yz} = J_{yz_{I}} + J_{yz_{II}} - J_{yz_{III}} =$$

$$= 0 + A_{I}(\zeta_{c_{I}} - \zeta_c)(\eta_{c_{I}} - \eta_c) + 0 + A_{II}(\zeta_{c_{II}} - \zeta_c)(\eta_{c_{II}} - \eta_c)$$

$$- 0 - A_{III}(\zeta_{c_{III}} - \zeta_c)(\eta_{c_{III}} - \eta_c) = \boxed{0.216\, a^4}$$

Zadanie 8.2 (c.d.)





$$J_1 = 0.718\,a^4$$

$$J_2 = 0.265\,a^4$$

$$\alpha_0 = 36{,}5^\circ$$